Analytical Prediction of the Incompressible Turbulent Boundary Layer with Arbitrary Pressure Distribution

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The boundary-layer equations for incompressible two-dimensional turbulent flows with arbitrary pressure distributions are solved by the method of weighted residuals. The method is capable of predicting skin friction, boundary-layer integral parameters, and velocity profiles. The analytical technique contains a theoretical convergence criterion and enables computations to be made to any desired accuracy subject to the accuracy of the selected turbulence model. To obtain solutions a mixing length model for the turbulent shear description is used in conjunction with initial profile data modified by orthogonal functions. Analytical results are compared with three experimental cases: the Schubauer and Klebanoff flow over an airfoil-like body which includes favorable and adverse pressure gradients and leads to separation, the equilibrium flow with an adverse pressure gradient of Bradshaw, and the favorable pressure gradient equilibrium flow of Herring and Norbury. In all three cases agreement is satisfactory. The method is sufficiently rapid to permit its use as a subprogram in analyses of broader extent.

Nomenclature

A_{ij} = coefficient matrix defined by Eq. (15)
B_{ij} = coefficient matrix defined by Eq. (16)
C_j = x-dependent variable in approximating function
C'_j = dC_j/dx
C_f = skin-friction coefficient, $\tau_w/\frac{1}{2}\rho U_\infty^2$
F = empirical decay parameter defined by Eq. (18)
g_i = turbulent shear integral defined by Eq. (12)
h_i = set of linearly independent weighting functions of u
h'_i = dh_i/du
h''_i = d^2h_i/du^2
l = turbulent mixing length defined by Eq. (18)
L = characteristic length
N = number of terms in approximating function and order of solution
P_j = Legendre polynomials
P'_j = dP_j/du
Re = reference Reynolds number, U_rL/ν
Re_{δ^*} = displacement thickness Reynolds number, $\left(\frac{U_\infty}{\nu}\right)\int_0^\infty (1-u)dy$
Re_θ = momentum thickness Reynolds number, $\left(\frac{U_\infty}{\nu}\right)\int_0^\infty u(1-u)dy$
u = dimensionless velocity, U/U_∞
u^+ = dimensionless velocity, $u/(C_f/2)^{1/2}$
u_i^+ = data representing experimentally measured velocity profile
U = mean velocity component in direction of x coordinate
$U_{\rm def}$ = defect velocity, $(1-u)(2/C_f)^{1/2}$
U' = velocity fluctuation from mean component in direction of x coordinate
v = dimensionless velocity, V/U_∞
V = mean velocity component in direction of y coordinate
V' = velocity fluctuation from the mean component in direction of y coordinate
W_i = set of linearly independent weighting functions of y
x = coordinate direction parallel to surface
y = coordinate direction normal to surface
y^+ = dimensionless coordinate normal to surface, $(yU_\infty/\nu)(C_f/2)^{1/2}$
y_i^+ = data representing experimentally measured velocity profile
α = empirical constant in turbulent shear model
Δ = thickness parameter, $\Delta = \delta^*/(C_f/2)^{1/2}$
$\Delta u^+, \Delta y^+$ = incremental differences of measured velocity profile data
Θ = transformed dependent variable defined by Eq. (7)
$\tilde{\Theta}$ = approximating function representing transformed dependent variable Θ
μ = dynamic viscosity of fluid
ν = kinematic viscosity of fluid
ρ = mass density of fluid
τ = shear stress defined by Eq. (4)
ϕ_j = u-dependent variable in approximating function
$(\ )_w$ = evaluated at the wall ($y = 0$)
$(\ )_r$ = reference value
$(\ )_x$ = derivative with respect to x
$(\ )_y$ = derivative with respect to y
$(\ )_o$ = initial condition
$(\ )_\infty$ = freestream condition

Presented as Paper 69-379 at the AIAA 2nd Advanced Marine Vehicles and Propulsion Meeting, Seattle, Wash., May 21–23, 1969; submitted April 14, 1969; revision received September 25, 1969. This work was supported in part by NASA and by U.S. Army Research Office, Durham, S.C.

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Introduction

A GENERAL analysis of the turbulent boundary layer is fundamental to the development of advanced hydrodynamic marine vehicles. In choosing a method of boundary-layer prediction for use in design, it is important to realize that conflicting requirements may arise. For example, a routine calculation procedure may be desired that balances cost versus desired accuracy. On the other hand, it may be necessary to have a method that can easily be extended to fill some future calculation requirement. Since the turbulent boundary-layer prediction usually forms only a part of the complete design calculation, the choice of a particular method must be made with the aim of balancing the boundary-layer proportion of the total analysis, including calculation time, against the over-all accuracy of the final answer.

Considerable analytical work has been done in recent years on turbulent boundary-layer prediction methods. The methods of Bradshaw and Ferriss,[1] Cebeci and Smith,[2] and Mellor and Herring[3] represent some of the more successful finite difference approaches. These approaches may often require more computer time and storage per calculation case than is usually acceptable in design or production run applications. A second class of techniques are the integral methods such as those proposed by McDonald and Camarata[4] and Hirst and Reynolds.[5] Although the integral methods are usually fast and reasonably accurate, they are often difficult to extend to include more complicated effects such as transverse curvature or boundary-layer suction. The present work describes an analysis based upon the method of weighted residuals, or MWR, and combines the ease of analytical extension of finite difference methods with the computation speed of integral methods. The concepts behind the MWR encompass both the finite difference and integral methods and, when applied to the incompressible two-dimensional turbulent boundary-layer momentum and continuity equations, can be used with a minimum of empirical input to describe flows involving arbitrary, known pressure distributions. Convergence of this method has been proved in many instances (e.g., see Finlayson and Scriven[6]) and has been demonstrated for the laminar boundary-layer equations by Bethel and Abbott.[7] The accuracy of the approximate solution may be improved by increasing the order of approximation N, and unnecessary accuracy limits are not forced upon the user because of such mathematical considerations as numerical stability (e.g., in explicit finite difference methods), etc. For the turbulent boundary-layer problem the level of approximation is governed by the model describing the variation of the turbulent shear. Because the mixing length concept has proved to yield to unusually diverse extensions beyond the original physical arguments employed in its formulation (e.g., compressible flows, transverse curvature and transpiration), the present analysis employs a mixing length model, patterned after the formulation of Michael, Quemard, and Durant,[8] to describe this variation.

Method of Weighted Residuals

The method of weighted residuals has been successfully applied to laminar boundary layers in steady flow by Abbott and Bethel,[9] unsteady flow by Koob and Abbott,[10] compressible separated flows by Crawford and Holt,[11] and Nielsen, et al.[12] and to turbulent flows by Deiwert and Abbott.[13] Essentially, the MWR is an approximation technique that belongs to the same general class of procedures as the quasi-difference (marching integration and cross-stream integration) finite difference methods. The method consists of two steps. First, the variation of the dependent variable is described so as to satisfy approximately the given differential equation and boundary conditions. Second, the residual or error resulting from the approximation is multiplied by a suitable weighting function and the weighted residual is then distributed in a prescribed way so that the error at any point in the solution domain decreases as the complexity of the approximating function increases. The inherent appeal of this class of solution techniques is the theoretical possibility that a convergence criterion exists and thus that the solution error can always be reduced without limit. In the present work the MWR is applied to the turbulent boundary-layer equations by performing these two steps. Solutions by this analysis are then compared with existing experimental data and existing solutions by other methods.

Analysis

The boundary-layer equations for steady, two-dimensional, incompressible turbulent flow can be written as follows:

$$UU_x + VU_y - U_\infty U_{\infty x} - (1/\rho)\tau_y = 0 \qquad (1)$$

$$U_x + V_y = 0 \qquad (2)$$

The boundary conditions are

$$U(x,0) = 0,\ V(x,0) = 0,\ U \rightarrow U_\infty \text{ as } y \rightarrow \infty \qquad (3)$$

with appropriate information being given at $x = x_0$ to provide an initial condition. The turbulent shear stress is given by

$$\tau = \mu(\partial U/\partial y) - \rho\overline{U'V'} \qquad (4)$$

The MWR is applied to the boundary-layer equations by multiplying Eq. (1) by a set of linearly independent weighting functions $W_i(y)$ and integrating the resulting equation over the boundary layer (i.e., from 0 to ∞). Formally, the following result is obtained [after combining Eqs. (1) and (2)]:

$$\int_0^\infty W_i(y)\left\{2UU_x + [UV]_y - U_\infty U_{\infty x} - \frac{1}{\rho}\tau_y\right\} dy = 0 \qquad i = 1 \text{ to } N \quad (5)$$

The relationship of Eq. (5) to various well-known analytical techniques is discussed by Abbott, et al.[14]

Equation (5) can be put into a form suitable for numerical solution by normalizing the variables and by employing a coordinate transformation. First, the dependent variables U and V are normalized with respect to the (known) free-stream velocity U_∞ so that

$$\left.\begin{aligned} u &\equiv U(x,y)/U_\infty(x) \\ v &\equiv V(x,y)/U_\infty(x) \end{aligned}\right\} \qquad (6)$$

Second, the coordinates are transformed by changing the variable of integration from y to u and the dependent variable u to $\Theta(x,u)$. The new dependent variable, Θ, is the inverse slope and is defined in normalized form by

$$\Theta(x,u) \equiv \frac{U_\infty/\nu}{Re^{1/2}(\partial u/\partial y)} \qquad (7)$$

where $Re = U_rL/\nu$ and L and U_r are an arbitrary reference length and velocity, respectively. This transformation achieves two things: it allows the skin friction to be directly related to the unknown parameters of the analysis, and it changes the limits of integration from a semi-infinite interval (0 to ∞) to a finite interval (0 to 1). To accommodate the new variables a new weighting function $h_i(u)$ replaces the $W_i(y)$.

The final step in applying the MWR to Eq. (5) is to approximate the dependent variable in some general way. Details of the criteria used in the selection of both the approximating and weighting functions are described in Refs. 13 and 14. With the preceding transformation the dependent variable $\Theta(x,u)$ may be approximated by

$$\Theta(x,u) \approx \tilde{\Theta}(x,u) = \sum_{j=1}^{N} C_j(x)\phi_j(u) \qquad (8)$$

where the function $\phi_j(u)$ must be a set of linearly independent functions which satisfy the boundary conditions on u. The form for $\phi_j(u)$ chosen for the present investigation is given by

$$\phi_j(u) = P_{j-1}(2u - 1)\phi_0(u)/(1 - u) \qquad (9)$$

where

$$\phi_0(u) = (1 - u)\Theta(x_0,u)/\Theta(x_0,0) \qquad (10)$$

The quantity $P_{j-1}(2u - 1)$ is the usual family of Legendre polynomials with arguments varying between plus and minus unity. The term $(1 - u)$ in the denominator of Eq. (9) forces the boundary condition at the outer edge of the boundary layer to be satisfied. The factor $\phi_0(u)$ represents the initial profile for Θ and is used to satisfy the initial condition.‡ The N unknown parameters $C_j(x)$ are then determined so that Eqs. (5) are satisfied.

Rewriting Eqs. (5) in terms of the transformed variables described by Eqs. (6–8) gives the basic MWR equations for the present analysis:

$$\frac{dC_j}{dx}\int_0^1 h_i u\phi_j du + \frac{dU_\infty/dx}{U_\infty} C_j \int_0^1 [h_i u - (1 - u^2)h'_i] \times \phi_j du + \frac{U_\infty/\nu}{Re^{1/2}}\left[h'_i(0)\frac{C_f}{2} + \frac{g_i}{Re^{1/2}}\right] = 0 \quad (11)$$

i and $j = 1$ to N

Details of the development of Eq. (11) are given in Refs. 13 and 14. Briefly, the $[UV]_y$ term in Eq. (5) has been eliminated by letting $h_i(1) = 0$ and the τ_y/ρ term has been integrated by parts to yield the last term in brackets in Eq. (11). The quantities g_i are called shear integrals and are defined in terms of physical variables as

$$g_i = \frac{C_f}{2}\frac{Re^{1/2}}{U_\infty}\int_0^\infty h''_i \frac{\tau}{\tau_w}\frac{\partial U}{\partial y} dy \quad (12)$$

where $i = 1, 2, \ldots, N$. In addition to the approximating function the manner in which the physical description of turbulence is introduced into the shear integrals and the selection of a family of weighting functions determines to a great extent the success of analyses using Eqs. (11).

Solution

Weighting Function and Turbulence Model

In order to solve Eq. (11) it is necessary to choose a set of weighting functions $h_i(u)$ and a turbulence model for evaluating g_i. Discussing first the weighting functions, it is important to realize that although theoretically any set of linearly independent functions may be chosen,§ it is practical to make a selection that leads to a well behaved analytical or numerical result. For the present case, it is useful to choose

$$h_i(u) = [1 - u]P_{i-1}(2u - 1) \quad (13)$$

This choice for h_i has two advantages. First, the combination of Eqs. (9) and (13) in Eq. (11) removes the singularity introduced by the term $(1 - u)$ needed to satisfy the boundary condition as $u \rightarrow 1$. Second, the evaluation of the integrals in Eq. (11) involving the product of the two orthogonal functions composed of Legendre polynomials lead to well-behaved coefficient matrices for numerical computation—an improvement over earlier works.[13,14]

Substitution of Eq. (13) into Eq. (11) yields the following result:

$$A_{ij}C' + \frac{dU_\infty/dx}{U_\infty} B_{ij}C_j + \frac{U_\infty/\nu}{Re^{1/2}}\left[h'_i(0)\frac{C_f}{2} + \frac{g_i}{Re^{1/2}}\right] = 0 \quad (14)$$

where

$$A_{ij} \equiv \int_0^1 h_i u \phi_j du = \int_0^1 P_{i-1}(2u - 1)P_{j-1}(2u - 1)u\phi_0 du \quad (15)$$

‡ This choice of ϕ_0 is not essential to the method but is used here to assure the satisfaction of all initial conditions. Because the equations are integrated across the layer it is necessary only to satisfy certain integral initial conditions and any characteristic representation that meets these requirements may be used.

§ As pointed out by J. D. Murphy of the NASA Ames Research Center, in a private communication, there does exist one set of weighting functions that are orthogonal to $u\phi_j$. For this singular case the coefficient matrix A_{ij} is null and no solution can be obtained. This is unlikely to pose a problem in actual practice since there are a large number of sets of weighting functions from which to choose.

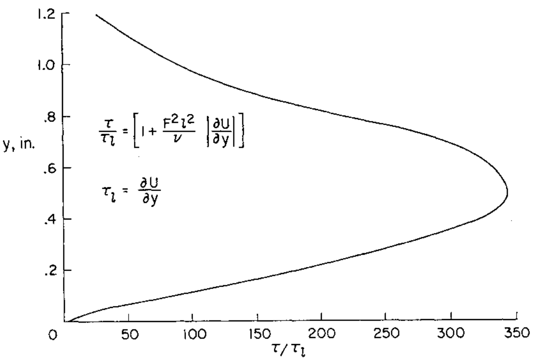

Fig. 1 Ratio of total shear to laminar shear for Bradshaw flow at $x = 1.917$ ft.

and

$$B_{ij} \equiv \int_0^1 [h_i u - (1 - u^2)h'_i]\phi_j du$$

$$= \int_0^1 [(1 + 2u)P_{i-1}(2u - 1) - (1 - u^2) \times P'_{i-1}(2u - 1)]\, P_{j-1}(2u - 1)\phi_0 du \quad (16)$$

Note that for a given initial distribution (ϕ_0), the matrix coefficients A_{ij} and B_{ij} are constants. Furthermore, C_f can be expressed directly in terms of C_j. U_∞ is a specified function of x, so that Eqs. (14) are sufficient to determine the $C_j(x)$ when the shear integrals g_i are specified. Thus, the initial distribution ϕ_0 is sufficient to define the A_{ij} and the B_{ij} and similarly an initial value for C_f is sufficient to define initial values for the $C_j(x_0)$. While approximations have been made to reduce the governing Eqs. (1) and (2) to the simple form given by Eq. (14), the empirical turbulent model appears only in the shear integral term, g_i, and the generality of the method is preserved.

Turning now to the turbulence description, the mixing length model suggested by Michel, Quémard, and Durant[8] was slightly modified and used in the form

$$\tau = \mu\left[1 + \frac{F^2 l^2}{\nu}\left|\frac{\partial U}{\partial y}\right|\right]\frac{\partial U}{\partial y} \quad (17)$$

where

$$F = 1 - \exp\left(-\alpha\frac{yU_\infty}{\nu}\frac{C_f}{2}\right)$$

$$l = 0.085\delta \tanh\left(\frac{0.41y}{0.085\delta}\right) \quad (18)$$

The parameter α in the definition of F may be chosen so that the gradient of skin-friction coefficient at the initial condition, $(dC_f/dx)_{x_0}$, is reproduced or may be taken as unity when this gradient is unknown. A typical distribution of the shear stress τ given by Eqs. (17) and (18) is shown in Fig. 1.

Solution Technique

To solve Eqs. (14) four steps must be taken: 1) Specification of external flow conditions, 2) specification of initial conditions, 3) selection of a method for solving the differential Eqs. (14), and 4) conversion of the dependent variables, $C_j(x)$, to boundary-layer parameters.

Specification of external flow conditions will not be considered here. It is assumed that sufficient information is given to compute U_∞ and dU_∞/dx and certain flow constants such as Reynolds number (Re), Reynolds number per unit length (Re/L) and the characteristic length (L) or velocity (U_r).

At the initial station x_0, it is necessary to specify the distribution for $\phi_0(u)$, a value for skin-friction coefficient, C_{f_0},

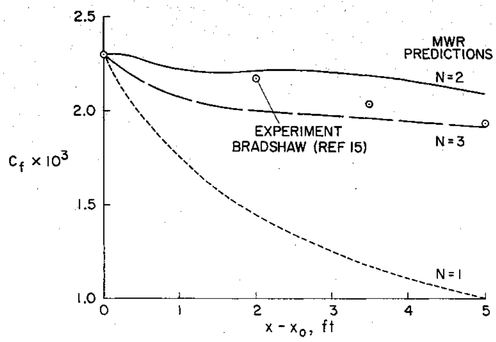

Fig. 2 MWR convergence for Bradshaw skin-friction coefficient variations; x_0 = 1.917 ft.

and any further initial data that may be required in the empirical descriptions for the turbulent shear variation.

The initial distribution, ϕ_0, is defined by Eq. (10) and may be written in the form

$$\phi_0(u) = [(1 - u)/(du^+/dy^+)]_0 \tag{19}$$

If sufficient velocity profile data (u vs y) are available at x_0, the gradient du^+/dy^+ may be approximated by suitable numerical techniques. In many instances it is possible to augment initial profile data with the law of the wall or law of the wake. In the present work, velocity gradients were approximated by constructing linear slopes (secants) between adjacent data points [i.e., $du^+/dy^+ \approx \Delta u^+/\Delta y^+ = (u_{i+1}{}^+ - u_{i-1}{}^+)/(y_{i+1}{}^+ - y_{i-1}{}^+)$]. Since the function $\phi_0(u)$ is always integrated in the present analysis, the linear approximation is sufficient for engineering calculations.

The initial coefficients, $C_j(x_0)$, are evaluated from the initial skin-friction coefficient value. The approximate expression for Θ given by Eq. (8) becomes exactly the initial data at the initial station x_0 and is written below using the definition of $\phi_j(u)$ given by Eqs. (9) and (10)

$$\Theta(x_0,u) = \sum_{j=1}^{N} C_j(x_0)\,\frac{P_{j-1}(2u - 1)}{1 - u}\,(1 - u)\,\frac{\Theta(x_0,u)}{\Theta(x_0,0)}$$

or, rearranged

$$\Theta(x_0,0) = \sum_{j=1}^{N} C_j(x_0)P_{j-1}(2u - 1)$$

Since the $\Theta(x_0,0)$ is a constant and the $P_{j-1}(2u - 1)$ are linearly independent functions of u it is obvious that the $C_j(x_0)$ must be identically zero for values of j greater than unity. For $j = 1$, the Legendre polynomial P_0 is unity for all values of u and so the initial coefficients are defined

$$C_1(x_0) = \Theta(x_0,0)$$

$$C_j(x_0) = 0 \text{ for } j > 1$$

From the definition of $\Theta(x,u)$ it can be shown that

$$\Theta(x_0,0) = 1/(C_{f0}/2)Re^{1/2}$$

Thus, the following relationships exist for the determination of the initial C_j

$$\left.\begin{aligned} C_1(x_0) &= 2/C_{f0}Re^{1/2} \\ C_j(x_0) &= 0 \text{ for } j > 1 \end{aligned}\right\} \tag{20}$$

The unknown constant α in the turbulent shear distribution is evaluated by iterating to find the value of α that will yield the values of $C'_j(x_0)$ from Eq. (14) which reproduce $dC_f/dx|_{x_0}$. Values for α should be near unity and, if $dC_f/dx|_{x_0}$ is unknown, may be approximated by unity.

With the initial conditions and external flow conditions specified, the governing Eqs. (14) can be solved by any suitable numerical integration scheme. The present results were obtained using a fourth order Adams-Moulten predictor-corrector technique.

Using the solution to the differential equations for the $C_j(x)$, the description of $\Theta(x,u)$ is complete. Employing Eq. (8), it is possible to compute the skin-friction coefficient C_f, Reynolds numbers based on such lengths as momentum and displacement thickness, Re_θ and $Re_\delta{}^*$, and velocity distributions, $u(y)$. Expressions for these quantities are

Skin friction:

$$\frac{C_f(x)}{2} = \left[Re^{1/2}\sum_{j=1}^{N}(-1)^{j-1}C_j(x)\right]^{-1} \tag{21}$$

Thickness parameters:

$$Re_\theta(x) = Re^{1/2}\int_0^1 u(1 - u)\tilde{\Theta}(x,u)du \tag{22}$$

$$Re_\delta{}^*(x) = Re^{1/2}\int_0^1 (1 - u)\tilde{\Theta}(x,u)du \tag{23}$$

Velocity profiles:

$$y(x,u) = Re^{1/2}\,\frac{\nu}{U_\infty}\int_0^u \tilde{\Theta}(x,u)du \tag{24}$$

Results and Discussion

It is expedient to demonstrate the capability of the MWR to converge as the order of solution is increased and to indicate the level of accuracy expected with the present mixing length model before comparing results from the MWR with results from other analytical methods and with experimental data. Illustrated in Figs. 2 and 3 are skin friction and velocity profile results obtained by first, second- and third-

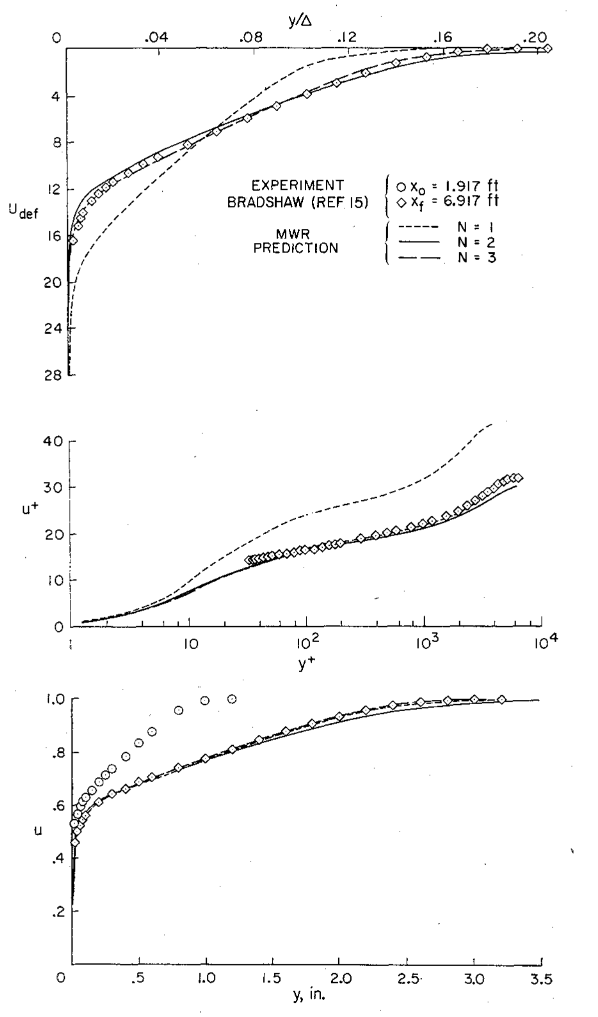

Fig. 3 MWR convergence for Bradshaw velocity profiles at $x = x_f$ = 6.917 ft.

($N = 1$, 2, and 3) order MWR solutions for the equilibrium flow of Bradshaw.[15] The skin-friction results in Fig. 2 indicate the first-order solution underpredicts C_f, the second-order solution slightly overpredicts C_f, and the third-order solution slightly underpredicts C_f. Assuming the experimental data are correct it is clear that the desired solution lies somewhere between the second- and third-order solutions. The inherently inadequate empirical description of the turbulent shear variation precludes ever reaching the solution exactly, but the possibility exists to improve the solution as improved descriptions for the turbulent shear are developed.

The results shown in Fig. 3 illustrate the capability of the MWR to describe velocity profiles. Included in this figure are velocity profile descriptions for the final data station ($x_f = 6.917$ ft) and the experimentally measured velocity distribution for both the initial and final data station. The profile predictions are presented in three different coordinate systems¶ and are all perturbed forms of the initial profile which was used to define ϕ_0 in the approximation function. Upon close examination, improved agreement with the experimentally measured distribution can be seen for increasing orders of solution. It would be difficult to show any closer agreement that might be obtained by higher order solutions.

The results in Figs. 2 and 3 serve to demonstrate that second- or third-order solutions are sufficient for most engineering design calculations. Due to the approximations present in the turbulent shear description, it is doubtful if any further gain could be made at this time by employing higher-order solutions.

To demonstrate the capability of the MWR as a turbulent boundary-layer prediction technique the results for skin-friction coefficient, momentum thickness and velocity distribution are presented for three typical flow cases. These

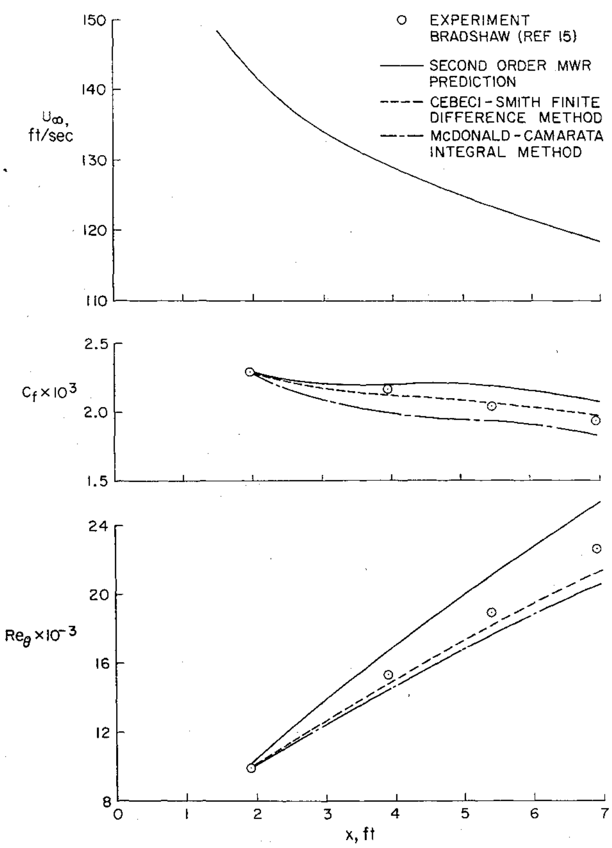

Fig. 4 Predicted results for the equilibrium flow of Bradshaw.

¶ Law of the wall, velocity defect, and physical coordinates.

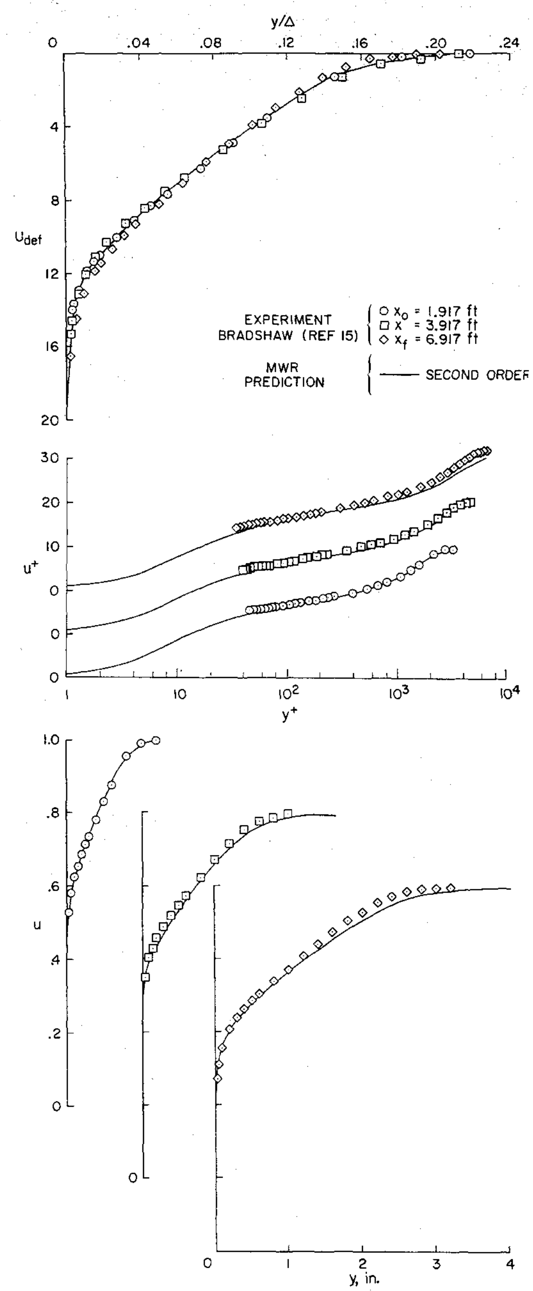

Fig. 5 Velocity profile predictions for the Bradshaw equilibrium flow.

flows represent a variety of external conditions and are considered here to provide a general test of the method. The results were computed by a two parameter ($N = 2$) method, the lowest order containing sufficient turbulence information, and are compared both with experimental data and predictions typical of finite difference and integral analyses. Considered here were the finite difference method of Cebeci and Smith[3] and the integral method of McDonald and Camarata,[5] generally accepted as two of the better prediction techniques available at this time. Like the MWR, these other methods rely on models describing the details of the turbulent shear variation. The Cebeci-Smith method employs an eddy viscosity model and the McDonald-Camarata a mixing length concept, both different from the model used in the present study.

Bradshaw Equilibrium Flow

Figures 4 and 5 show results for the equilibrium flow of Bradshaw[15] for an adverse pressure gradient. The solutions

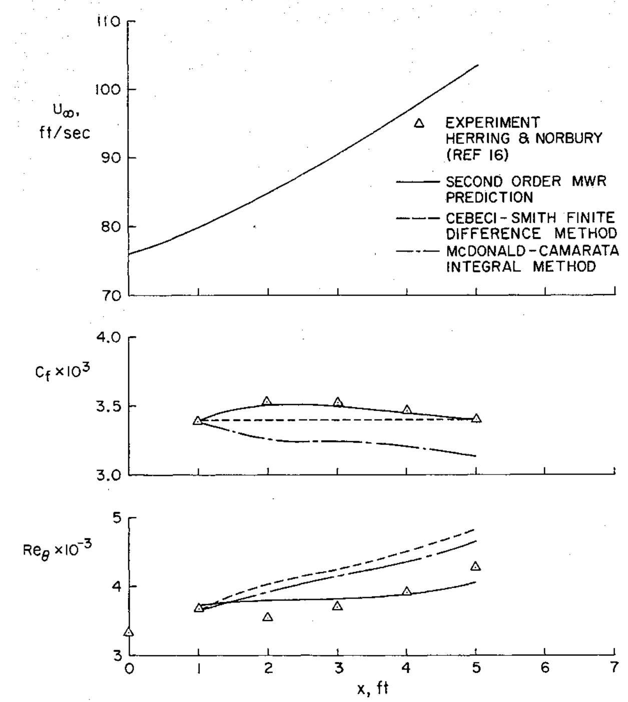

Fig. 6 Predicted results for the equilibrium flow of Herring and Norbury.

were begun at an initial point of $x_0 = 1.917$ ft and concluded at $x_f = 6.917$ ft. The MWR results for C_f and Re_θ lie slightly above the data and the results from the Cebeci-Smith and McDonald-Camarata methods. All three methods give solutions of the same over-all quality; the McDonald-Camarata results are slightly low and the Cebeci-Smith results somewhere in between the other two.

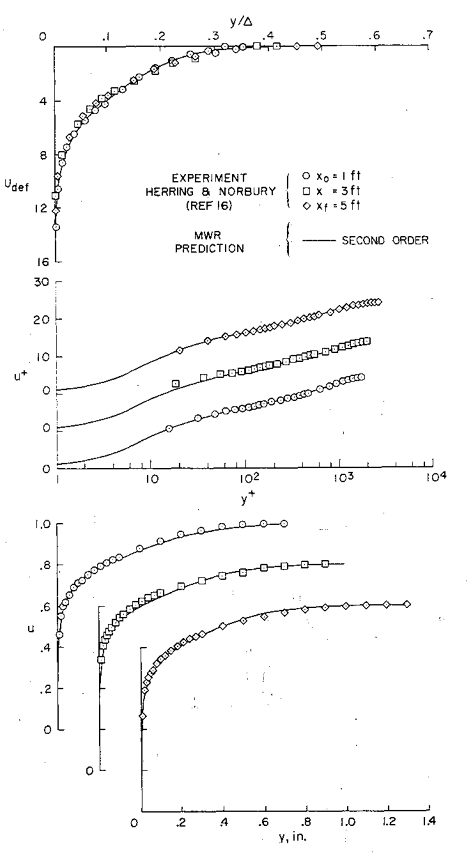

Fig. 7 Velocity profile predictions for the Herring and Norbury equilibrium flow.

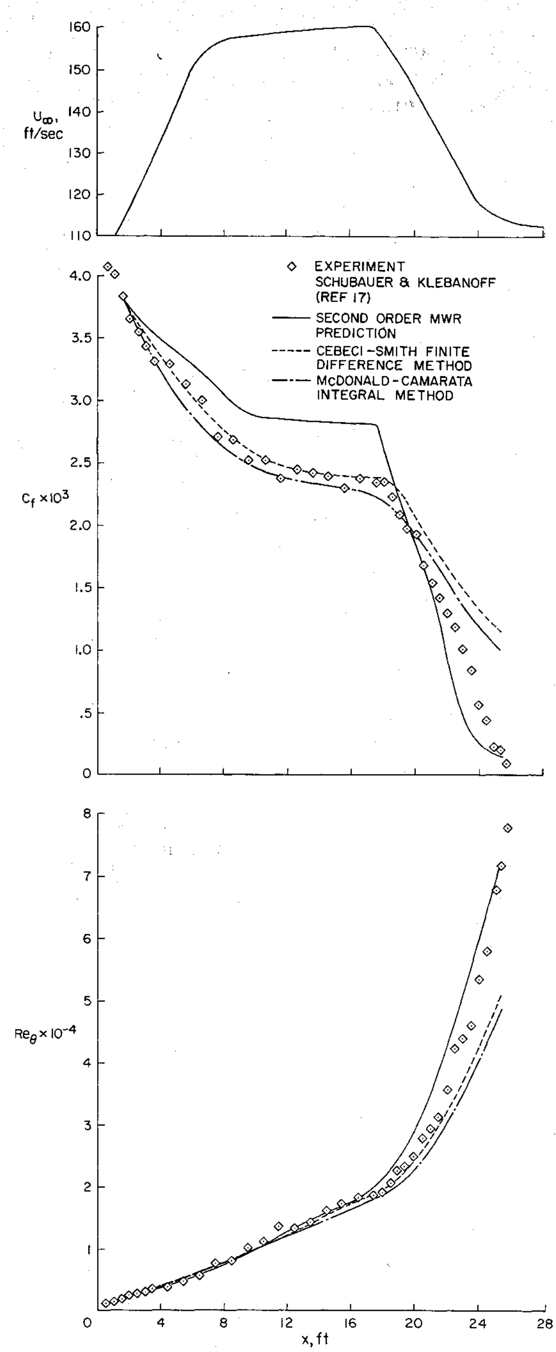

Fig. 8 Predicted results for the Schubauer and Klebanoff flow over an airfoil-like body.

Figure 5 shows velocity profiles for the initial, intermediate, and final x values. The agreement between the computed and measured distributions is very good in each of the coordinate systems. Note especially the collapse of the profiles into a single curve in the velocity defect coordinates. The characteristic feature of equilibrium flows is adequately described by the MWR solution.

Herring and Norbury Equilibrium Flow

Figures 6 and 7 show results for the equilibrium flow of Herring and Norbury[16] for a favorable pressure gradient. The solutions were begun at an initial point of $x_0 = 1$ ft and concluded at $x_f = 5$ ft. The MWR solutions for both C_f and Re_θ clearly agree quite well with the experimental data and are in better over-all agreement than either the Cebeci-Smith or McDonald-Camarata results. The velocity distributions in Fig. 7 again show excellent agreement

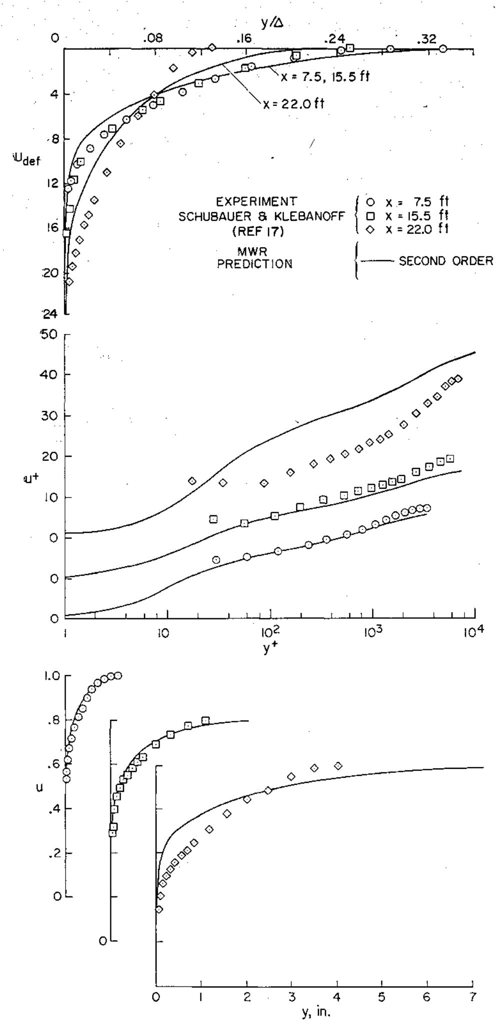

Fig. 9 Velocity profile predictions for the Schubauer and Klebanoff flow over an airfoil-like body.

with the experimentally measured distributions. These profiles are presented for the initial, intermediate and final x values in the three common coordinate systems. Again, note the collapse of the profiles to a single curve in the defect coordinates.

Schubauer and Klebanoff Flow

Figure 8 and 9 show results for the Schubauer and Klebanoff[17] flow over an airfoil-like body. This case represents flow under first a favorable then an adverse pressure gradient leading ultimately to separation at $x = 25.77$ ft. The solutions were begun at $x_0 = 1.5$ ft and concluded at $x_f = 25.4$ ft. The present predictions for C_f were high in the favorable pressure gradient region and slightly low in the adverse pressure gradient region. This disparity is attributed to the model chosen for the turbulent shear since other formulations used by the authors for the turbulent shear yielded predictions similar to those of Cebeci-Smith and McDonald-Camarata. The MWR solution does come closer to predicting the separation point than either the Cebeci-Smith or McDonald-Camarata solutions, and looking at the results for Re_θ it is seen that the best over-all agreement with the experimental data is given by the MWR. The authors hesitate, however, to draw any firm conclusions from this due to the uncertainty of the two dimensionality of the data toward the trailing edge.

Velocity profiles are shown in Fig. 9 in the three coordinate systems for $x = 7.5$ ft, 15.5 ft, and 22.0 ft. While the agreement with the experimentally measured distributions is not exact these profiles do yield the correct values for the thickness parameters and skin friction and do indicate the proper gross behavior (i.e., a thickening of the boundary layer). As discussed in Bethel and Abbott,[7] further considerations must be made in specifying approximating functions that are to be valid in regions near separation. These considerations have not been made in the present work.

Concluding Remarks

A method has been developed for analytically predicting incompressible turbulent boundary-layer behavior. To obtain solutions it is necessary to specify initial values for skin friction and velocity distribution, as well as the external flow conditions. All empirical constants are associated with the arbitrarily selected turbulent shear model which may be as simple or elaborate as desired. In the present paper, a mixing length description for the turbulent shear was employed and the method was used to calculate the behavior of three typical boundary-layer flows. While it is recognized that the shear description can always stand further improvement, on the basis of the predicted results for skin friction, momentum thickness and velocity distribution, it can be concluded that the present method provides a satisfactory technique for routine engineering design. Of equal importance, the present method not only retains the simplicity of the integral methods, but it also contains the generality of the finite difference techniques and can be easily extended to include more complicated effects that may be desired.

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